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UNIQUE MAXIMUM PROPERTY OF THE

STIRLING NUMBERS OF THE SECOND KIND

by

W. E. Bleick and Peter C. C. Wang

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Unique Maximum Property of the
Stirling Numbers of the Second Kind

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ABSTRACT:

Letting $f(n)$ and $\ell(n)$ the first and last maximum of the graph $S(n,k); k = 1, 2, \ldots, n$, Kanold [J. Reine Angew. Math 230(1968), 211-212] shows that, for sufficiently large n, $n/\log n < f(n) \leq \ell(n) \leq n\, h(n)/\log n$ with $h(n)$ subject only to $h(n) \to \infty$ as $n \to \infty$. This result was subsequently improved by Harborth [j. Reine Angew. Math 230(1968), 213-214] to yield $\lim_{n\to\infty} f(n)n^{-1}\log n = \lim_{n\to\infty} \ell(n)n^{-1}\log n = 1$. Together with Harper's result [Ann. Math. Stat. 38(1968), 410-414], it is concluded that $S(n,k)$ have, asymptotically, a single maximum. Lieb [J. of Comb. Theory 5(1968), 203-206] shows that Stirling numbers of the second kind possess the property of Strong Logarithmic Concavity, and thus are unimodal. Dobson [J. of Comb. Theory 5(1968), 212-214 and Vol. 7(1969), 116-121] shows a similar result in a stronger form. However, the classical problem of establishing that $S(n,k)$ possess an "unique" maximum for all $n \geq 3$ remains unsolved. It is the purpose of this paper to provide the complete solution of this classical problem.

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Letting $f(n)$ and $\ell(n)$ the first and last maximum of the graph $S(n,k); k = 1, 2, \ldots, n$, Kanold [J. Reine Angew. Math 230(1968), 211-212] shows that, for sufficiently large n, $n/\log n < f(n) \leq \ell(n) \leq n\, h(n)/\log n$ with $h(n)$ subject only to $h(n) \to \infty$ as $n \to \infty$. This result was subsequently improved by Harborth [J. Reine Angew. Math 230(1968), 213-214] to yield $\lim_{n\to\infty} f(n)n^{-1}\log n = \lim_{n\to\infty} \ell(n)n^{-1}\log n = 1$. Together with Harper's result [Ann. Math. Stat. 38(1967), 410-414], it is concluded that $S(n,k)$ have, asymptotically, a single maximum. Lieb [J. of Comb. Theory 5(1968), 203-206] shows that Stirling numbers of the second kind possess the property of Strong Logarithmic Concavity, and thus are unimodal. Dobson [J. of Comb. Theory 5(1968), 212-214 and Vol. 7(1969), 116-121] shows a similar result in a stronger form. However, the classical problem of establishing that $S(n,k)$ possess an "unique" maximum for all $n \geq 3$ remains unsolved. It is the purpose of this paper to provide the complete solution of this classical problem.

I. Introduction

The Stirling numbers of the second kind $S(n,k)$ have come into renewed salience, primarily due to the fact that $S(n,k)$ is the number of partitions of an n-set into k disjoint nonempty subset and $S(n,k)$ is the number of distinct fields defined on a finite sample space with n elementary events to which each field contains exactly 2^k events [1]. Letting $f(n)$ and $\ell(n)$ the first and last maxima of the graph $S(n,k); k = 1, 2, \ldots, n$, Kanold [2] shows that, for sufficiently large n, $n/\log n < f(n) \le \ell(n) \le n\, h(n)/\log n$ with $h(n)$ subject only to $h(n) \to \infty$ as $n \to \infty$. This result was subsequently improved to yield

$$\lim_{n\to\infty} f(n)n^{-1} \log n = \lim_{n\to\infty} \ell(n)n^{-1} \log n = 1,$$

by Harborth [3]. Together with Harper's result [4], it is concluded that $S(n,k)$ have, asymptotically, a single maximum. Earlier Lieb [5] shows that Stirling numbers of the second kind possess the property of Strong Logarithmic Concavity, and thus are unimodal. Dobson [6, 7] shows a similar result in a stronger form. However, the classical problem of establishing that $S(n,k)$ possess an "unique" maximum for all $n \ge 3$ remains unsolved. It is the purpose of this paper to provide the complete solution of this classical problem.

II. Unique Maximum Property of $S(n,k)$.

In Riordan [8; p.43] has given the Taylor series

$$(1) \qquad \sum_{n=k}^{\infty} S(n,k) z^{n-k} = \prod_{j=1}^{k} (1-jz)^{-1},$$

convergent for $|z| < k^{-1}$, as a generating function for the Stirling numbers $S(n,k)$ of the second kind. The reciprocal transformation $z = w^{-1}$ converts (1) to the Laurent series

$$(2) \qquad \sum_{n=k}^{\infty} S(n,k) w^{-n} = \prod_{j=1}^{k} (w-j)^{-1},$$

convergent for $|w| > k$. The coefficient in the series (2) may be expressed as the contour integral

$$(3) \qquad S(n,k) = \frac{1}{2\pi i} \int_C \frac{w^{n-1} dw}{(w-1)(w-2)\dots(w-k)}$$

where the contour C encloses the singular points of the integrand. From (3) it follows that

$$(4) \qquad S(n,k-1) - S(n,k) = \frac{1}{2\pi i} \int_C \frac{w^{n-1}(w-k-1) dw}{(w-1)(w-2)\dots(w-k)} .$$

On evaluating (4) by the residue theorem we find

$$(5) \qquad S(n,k-1) - S(n,k) = \sum_{\ell=1}^{k} \lim_{w \to \ell} \frac{(w-\ell) w^{n-1}(w-k-1)}{(w-1)(w-2)\dots(w-k)} .$$

On using the Lagrange interpolation formula we interpret $S(n,k-1) - S(n,k)$ in (5) as the coefficient of v^{k-1} in the polynomial passing through the k ordinates

$$f(v) = v^n - (k+1)v^{n-1}$$

for $v = 1, 2, \ldots, k$. But this coefficient of v^{k-1} in the interpolating polynomial for the ordinates $f(1), f(2), \ldots, f(k)$ is proportional to the determinant

$$\begin{vmatrix} f(1) & f(2) & \ldots & f(k-1) & f(k) \\ 1 & 2^{k-2} & \ldots & (k-1)^{k-2} & k^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \ldots & (k-1) & k \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -k & -(k-1)2^{n-1} & \ldots & -2(k-1)^{n-1} & -k^{n-1} \\ 1 & 2^{k-2} & \ldots & (k-1)^{k-2} & k^{k-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \ldots & (k-1) & k \\ 1 & 1 & \ldots & 1 & 1 \end{vmatrix} .$$

The last $k-1$ rows of this determinant are the linearly independent rows of a Vandermonde determinant.

The first row of the determinant is a linear combination of the remaining rows only if $n = k = 2$. Hence, $S(n,k-1) - S(n,k) = 0$ only if $n = k = 2$. We have established the following Theorem: The Stirling numbers of the second kind $S(n,k)$ possess an "unique" maximum for all $n \geq 3$.

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